An introduction to a group theoretic approach to geometry.

By Jonathan Olds.

Goals:
During this short introduction I hope to explain the very rudimentary foundations of how one can look at geometry in a group theoretic way. I will describe how this can be done using Bachmann's axioms to form a group. Then I will go on to show various consequences of such axioms. I will pitch this essay at someone who has a minimal understanding of group theory.

Introduction:
Geometry was one of the first forms of mathematics studied by ancient civilizations. Euclid, who lived around 300 B.C., had already written many books about geometry, the contents of which most laypeople today take for granted, and seem to believe this is all that geometry is. But, contrary to popular belief, Geometry has progressed in the last few thousand years; the group theoretic approach being just a small part of the progress. The groups that I will form are called Bachmann groups; and Euclidean, hyperbolic, and elliptic geometries are all special cases of such groups.

Motivation for axioms:
First of all, we need to axiomatise geometry. Where should we begin? Well, we have to decide what sort of geometry we want. Should it be Euclidean, hyperbolic, elliptic, or something else? Would it be possible to encompass all such geometries? As a place to start let's consider only the continuous Euclidean plane in this section and see what we can derive.

Incidence and perpendicularity:
Lines are denoted by lowercase Roman letters, a,b,c,…
Points are denoted by uppercase Roman letters, A,B,C…
Incidence of a point A with a line g is denoted, A I g or g I A.
Perpendicular lines a,b are denoted, a $\perp$ b or b $\perp$ a.
A I g, A I h, B I h, B I g, will be denoted A,B I g,h.
And so on.

Mappings:
For all mappings α and β. Let $A\alpha\beta$ be the point A mapped through α and then through β. Likewise $b\alpha\beta$ be the line b mapped through α then β.

Perpendicular collineation:
A mapping from points to points and lines to lines such that incidence and perpendicularity are preserved.

Line reflections:
A line reflection a, is a bijection on the plane such that lines get mapped to lines, and points get mapped to points; it is a perpendicular collineation that preserves distance between points. The only fixed lines are a, and all lines perpendicular to a. The only fixed points are points incidence with a.
It will be denoted σ_a. S will denote the set of all line reflections.

Point reflections:
A point reflection P, is a bijection on the plane such that lines get mapped to lines, and points get mapped to points, it is a perpendicular collineation that preserves distance between points. The only fixed lines are lines incidence with P. The only fixed point is P. It will be denoted as σ_A.

Let ϕ be the relation from lines to line reflections. Take a line g, distance must be preserved and points not on this line must move perpendicular to g. If you were to have taken a point on g, then by definition this point does not move. This means that ϕ is a function, as everything is defined once you take a line. So take this function, : $x\varphi=\sigma_x$. Then ϕ is onto the set of all line reflections by definition. If $a \neq b$ then

Case 1. $a \not\perp b$

$\Rightarrow a\sigma_a = a \Rightarrow (a \perp b) \wedge (a \neq b) \Rightarrow a\sigma_b \neq a \Rightarrow \sigma_a \neq \sigma_b$

Case 2. $a \perp b$

take a point A : $(A \text{ I } a) \wedge (A \not\text{I}\, b) \Rightarrow A\sigma_a=A \wedge A\sigma_b \neq A \Rightarrow \sigma_a \neq \sigma_b$

$\therefore$ φ is a bijection, and there is an identification between lines and line reflections.
Let θ be the relation from points to point reflections. This also is a function, by a similar argument as above. So take this function : $A\theta=\sigma_A$. Then θ is onto the set of all point reflections by definition. If $A\neq B$ then $A\sigma_A=A$ but as $A\neq B \Rightarrow A\sigma_B\neq A$ $\therefore$ θ is the bijection, and there is an identification between points and point reflections.
This now allows us to treat the concepts of points and point reflections, and line and line reflections interchangeably.

Involutions:
An element β, of a group is an involution iff $\beta^2=1$ and $\beta\neq 1$.
If two involutions α,β such that $\alpha\beta$ is also an involution, will be written as $\alpha \mid \beta$.
Along the same lines of the notation for perpendicular and incidence we shall use similar abbreviations. E.g. $(\alpha \mid \beta) \wedge (\delta \mid \beta) \Rightarrow (\alpha, \delta \mid \beta)$ etc.

Isometry:
An Isometry is a sequence of line reflections. Isometries will be denoted as lowercase Greek letters if not denoted as a single line reflection. M will denote the set of all isometries.

Inner automorphism:
The innerautomorphism by an element γ on an element P, will be denoted as P^{γ}, i.e. $P^{\gamma} = \gamma^{-1} P \gamma$. It is also called the *transformation* of P by γ.

Invariant subset of a group:
A subset of a group that is closed under all innerautomorphisms by elements from the group is said to be an invariant subset of the group.
I.e. given a group M such that $S \subseteq M$ then $(\forall a \in S)(\forall \beta \in M) \Rightarrow (a^{\beta} \in S)$.

Clearly all line and point reflections are involutions.

The set of all isometries M form a group.
Proof:

M≠Ø (obvious)
Take a line reflection σ_a from M, then $\sigma_a\sigma_a \in M \Rightarrow 1 \in M$. (as σ_a is an involution)
If $\alpha,\beta \in M \Rightarrow$ there exists line reflections $\sigma_{\alpha 1},\ldots,\sigma_{\alpha k},\sigma_{\beta 1},\ldots,\sigma_{\beta n}$, : $\alpha=\sigma_{\alpha 1}\ldots\sigma_{\alpha k}$ and $\beta=\sigma_{\beta 1},\ldots,\sigma_{\beta n} \Rightarrow \alpha\beta= \sigma_{\alpha 1}\ldots\sigma_{\alpha k}\sigma_{\beta 1}\ldots\sigma_{\beta n} \Rightarrow \alpha\beta \in M$.
If $\alpha \in M \Rightarrow$ there exists line reflections $\sigma_{\alpha 1},..,\sigma_{\alpha k} \in M$: $\alpha=\sigma_{\alpha 1}..\sigma_{\alpha k}$
$\Rightarrow \delta = \sigma_{\alpha k},..,\sigma_{\alpha 1} \in M \Rightarrow (\alpha\delta=1) \wedge (\delta\alpha=1) \Rightarrow \alpha^{-1} \in M$.
And associativity is given by the fact that the composition function is associative.

(iii)
$$\boxed{\begin{array}{c}\text{Given } \alpha,\beta \text{ are involutions then,} \\ (\alpha\beta=\beta\alpha)\wedge(\alpha\neq\beta) \Leftrightarrow \alpha\beta \text{ is an involution } \Leftrightarrow \alpha \mid \beta\end{array}}$$

Proof:

Given α,β are involutions then…

If $(\alpha\beta = \beta\alpha) \wedge (\alpha \neq \beta) \Rightarrow \alpha\beta \neq 1$ is $\alpha \neq \beta$
$\Rightarrow (\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha\alpha\beta\beta = 1 \Rightarrow \alpha\beta$ is an involution.
If $\alpha\beta$ is an involution $\Rightarrow (\alpha\beta \neq 1) \wedge ((\alpha\beta)^2 = 1)$
$\Rightarrow \alpha\beta\alpha\beta = 1 \Rightarrow \alpha\beta\alpha = \beta \Rightarrow \alpha\beta = \beta\alpha$ as $\alpha^{-1} = \alpha$ and $\beta^{-1} = \beta$.
Also $(\alpha\beta \neq 1) \Rightarrow (\alpha \neq \beta)$.

(iv) $\boxed{\sigma_P^\alpha = \sigma_{P\alpha} \text{ and } \sigma_h^\alpha = \sigma_{h\alpha}}$

Proof:

(3) $A\gamma = B \Leftrightarrow A1\gamma\alpha = B\alpha \Leftrightarrow A\alpha\alpha^{-1}\gamma\alpha = B\alpha \Leftrightarrow (A\alpha)\gamma^\alpha = B\alpha$

$h\gamma = g \Leftrightarrow h1\gamma\alpha = g\alpha \Leftrightarrow h\alpha\alpha^{-1}\gamma\alpha = g\alpha \Leftrightarrow (h\alpha)\gamma^\alpha = g\alpha$

$A\sigma_h = B \Leftrightarrow (A\alpha)\sigma_{h\alpha} = B\alpha$, This is true as α is an Isometry. An easy way to justify this in your own mind, could be to take a piece of paper, draw a line h, with A reflected over it to B, then flip the piece of paper back and forth a few times. Lo and behold, Aα is reflected over hα to Bα. α was the process of flipping the piece of paper back and forth. To go the other way, just perform the flips in the exact reverse order so you get α^{-1} instead of α. Now, by 3, we have $A\sigma_h = B \Leftrightarrow (A\alpha)\sigma_h^\alpha = B\alpha$ but this implies that $(A\alpha)\sigma_h^\alpha = (A\alpha)\sigma_{h\alpha}$. For any point P, as α is a bijection, there exists an A, such that P=Aα. This means that $(\forall P)(P\sigma_h^\alpha = P\sigma_{h\alpha})$ which implies $\sigma_h^\alpha = \sigma_{h\alpha}$. Similar reasoning can show that $\sigma_P^\alpha = \sigma_{P\alpha}$.

This allows us to talk about elements of a group being multiplied together instead of a line being mapped to another line and then using that line to form a line reflection.

(v) $\boxed{\text{the set of all line reflections form an invariant subset of the isometries}}$

Proof:

The proof of this follows directly from the fact that $\sigma_h^\alpha = \sigma_{h\alpha}$ (iv). α Is a bijection from lines to lines which implies that $h\alpha$ is a line, meaning if S is the set of all line reflections, $\sigma_{h\alpha} \in S \Rightarrow \sigma_h^\alpha \in S$.

Now we can proceed by finding the equivalences between incidence and perpendicularity with group operations.

(vi) $\boxed{\begin{array}{c} P\mathrm{I}g \Leftrightarrow \sigma_P \mid \sigma_g \\ h \perp g \Leftrightarrow \sigma_h \mid \sigma_g \end{array}}$

Proof:

$P\mathrm{I}g \Leftrightarrow P\sigma_g = P$ By definition of a line reflection.

$\Leftrightarrow \sigma_{P\sigma_g} = \sigma_P$ Due to the bijection between point and point reflections.

$\Leftrightarrow \sigma_P^{\sigma_g} = \sigma_P$ By iv.

$\Leftrightarrow \sigma_g^{-1}\sigma_P\sigma_g = \sigma_P \Leftrightarrow \sigma_P\sigma_g = \sigma_g\sigma_P$

$\Leftrightarrow \sigma_P\sigma_g$ Is an involution by (iii) as $g \neq P$.

$\Leftrightarrow \sigma_P \mid \sigma_g$

$h \perp g \Leftrightarrow h\sigma_g = h$ By definition of a line reflection.

$\Leftrightarrow \sigma_{h\sigma_g} = \sigma_h \Leftrightarrow \sigma_h^{\sigma_g} = \sigma_h \Leftrightarrow \sigma_g^{-1}\sigma_h\sigma_g = \sigma_h \Leftrightarrow \sigma_h\sigma_g = \sigma_g\sigma_h$

$\Leftrightarrow \sigma_h\sigma_g$ Is an involution by (iii) as $g \neq h$.

$\Leftrightarrow \sigma_h \mid \sigma_g$

(vii)

If lines a,b meet at the point P then, $\sigma_a\sigma_b$ is a rotation around P of two times the directed angle from a to b.

That it is a rotation can easily be visualized in one's head by imagining a piece of paper, drawing two lines that intersect on it, flipping it over keeping the first line invariant, and then do the same with the second line. The point that the two lines intersect will have not moved, and the piece of paper will be the same side up as it was before the test. This means the only conclusion is that a rotation has been performed around the point of intersection. Now we take a point A, on the line a, such that A ≠ P. This implies that $A\sigma_a\sigma_b = A\sigma_b = A'$, then from a simple symmetry argument (fig1), this implies that A has been rotated twice the directed angle from a to b.

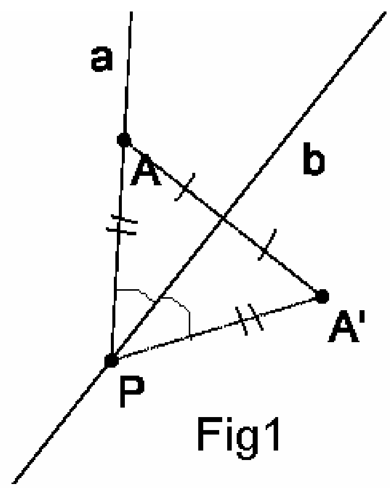
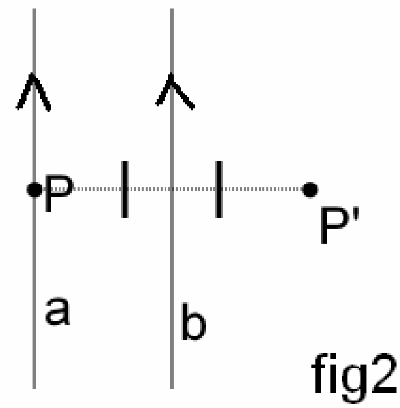


Fig1

If, the lines a and b didn't meet, then $\sigma_a\sigma_b$ is a translation perpendicular to a and b of twice the directed distance from a to b. That it is a translation can be visualized in a similar way to that when the lines intersect. For this visualization any line perpendicular to a and b will remain fixed, hence a translation has taken place. To show what this translation is, take a point P, on the line a. Then this implies that $P\sigma_a\sigma_b = P\sigma_b = P'$, again from a simple symmetry argument (fig2), we see that P has been moved twice the directed distance from a to b.



It's interesting to note that $(a,b,c,d\mathrm{I}P)\wedge(\sigma_a\sigma_b=\sigma_c\sigma_d)$ has the meaning that the directed angle between the lines a,b is equal to the directed angle between the lines c,d. This means that if you were to give me the lines a,b,c, I could measure the distance between a and b, then by measuring the same distance starting from c in direction of a to b, mark of a line d. This is just the line we want to produce $\sigma_a\sigma_b=\sigma_c\sigma_d$. By rearranging this equation we get $\sigma_d=\sigma_c\sigma_a\sigma_b$. Meaning, that for any three line reflections through a point, the product is a line reflection. The consequence of which is very important in Bachman groups. It is given the name of the three line reflection theorem, although soon we will take it to be an axiom, which can makes it confusing still calling it a theorem. Likewise by a similar argument there is a corresponding theorem concerning three lines all of which, have a common perpendicular. It states, given any three lines with a common perpendicular to a line g, there exists a line reflection equal to the product of the three lines, also being perpendicular to the line g. i.e.

(viii) $$\boxed{\begin{array}{c}(\forall c,a,b\mathrm{I}P)(\exists d\mathrm{I}P):\sigma_d=\sigma_c\sigma_a\sigma_b\\(\forall c,a,b\perp g)(\exists d\perp g):\sigma_d=\sigma_c\sigma_a\sigma_b\end{array}}$$

(ix) $$\boxed{(\forall P)(\exists g,h):(\sigma_P=\sigma_g\sigma_h)}$$

(vii) is the key in making line reflections the basic building block in our geometry. This is so, because in Euclidean geometry, any point has two mutually perpendicular lines that intersect it. Given any point P, and given these two lines g and h, (vii) implies that $\sigma_g\sigma_h$ is a rotation of two right angles, which is exactly half a rotation around P, in other words the point reflection σ_P. The consequence of which means we can define a point reflection as compositions of two mutually orthogonal line reflections. Thus enabling us to have the points defined by lines (ix).

Due to the identification between lines and line reflections, also, points and point reflections, this allows us to, without too much confusion, call point reflections points, and a line reflections lines. I.e. $\sigma_P = P$ and $\sigma_g = g$, “points” are point reflections and “lines” are line reflections. We shall now use this notion henceforth. Gathering up what we have done so far, we embark on defining a group that will describe what we have seen so far happening in Euclidean geometry.

We have seen by (ix) that we require only the set of lines, call the set S. S will generate a group having composition as its binary relation, call this group M. also, S is required to be an invariant subset of M, by (v). From (viii) & (vi) we have $(\forall c,a,b \mid P)(\exists d \mid P): d = cab$ and $(\forall c,a,b \mid g)(\exists d \mid g): d = cab$.

To this list we would also like to have lines between points. If these points were different we would like there to be a unique line joining them, and if they were the same, we would like more than one line to intersect that point. So we add $(\forall P,Q)(\exists g): P,Q \mid g$ and $P,Q \mid g,h \Rightarrow ((P = Q) \vee (g = h))$, (the existence of more than one line through the point when the two points are equal, is established, by the fact that a point can be defined as a composition of two perpendicular lines, where the point lies on each of the lines). Finally we would also like to remove the possibility of having trivial geometries, therefore we introduce, $(\exists g,h,j): (g \mid h) \wedge (j \nmid g) \wedge (j \nmid h) \wedge (j \nmid gh)$. That's it, we have our axioms.

Bachmann group:

Given a set S, and the group M generated by it. “Lines” being elements of S, and “points” being elements of M that are a product of two elements in S that are themselves involutions. Denoting lines by lowercase Roman letters, points by uppercase Roman letters.

If the following holds then M is a Bachmann group.

S is an invariant subset of M.
All elements in S are involutions.

Ax1 $(\forall P,Q)(\exists g): P,Q \mid g$
Ax2 $P,Q \mid g,h \Rightarrow ((P = Q) \vee (g = h))$
Ax3 $(\forall c,a,b \mid P)(\exists d \mid P): d = cab$
Ax4 $(\forall c,a,b \mid g)(\exists d \mid g): d = cab$
AxD $(\exists g,h,j): (g \mid h) \wedge (j \nmid g) \wedge (j \nmid h) \wedge (j \nmid gh)$

Consequences of the axioms:

To help visualize things in this section, remember that the *stroke relation*, $|$, can be thought of as either, the symbol that tells you that two lines are perpendicular, or a point and a line that are incidence. The equivalence relationship (iii) comes up regularly so keep it in the forefront of your mind.
Some very elementary properties of Bachmann groups are as follows, from now on I'll use them very frequently without mentioning them.

$$P = gh \Rightarrow \begin{array}{l} g \mid h \\ P \mid h \\ P \mid g \end{array}$$

$g \mid h \Leftrightarrow h \mid g$, $P \mid g \Leftrightarrow g \mid P$.

None of these are very profound, and they all follow directly from (iii) in the last section. For example, $P = gh \Rightarrow Ph^{-1} = Ph = g$, as g is an involution, it implies that Ph is also an involution, so, by (iii) $Ph = hP = g \Rightarrow P \mid h$.

(remark about transformations)
If, $g \mid h$, then $\forall \alpha \in M$ implies
$\exists P : P = gh \Rightarrow P^{\alpha} = \alpha^{-1} P \alpha = \alpha^{-1} gh\alpha = \alpha^{-1} g\alpha\alpha^{-1} h\alpha = g^{\alpha} h^{\alpha}$, but as S is an invariant subset of M, means g^{α}, h^{α} are both lines. Also, by the fact that $P^{\alpha} \neq 1$, and $(P^{\alpha})^2 = \alpha^{-1} P \alpha\alpha^{-1} P \alpha = 1$ means P^{α} is an involution, hence $g^{\alpha} \mid h^{\alpha}$. That is to say that the stroke relation is invariant under transformation. So remember that points get transformed into points, and lines into lines, with incidences and perpendicularity preserved. This is done in a 1-1 correspondence, as if
$\delta^{\alpha} = \beta^{\alpha} \Rightarrow \alpha^{-1} \delta \alpha = \alpha^{-1} \beta \alpha \Rightarrow \delta = \beta$.

It is now necessary to plow through some tedious proofs so we can use them at a later date. It does though give us a good chance to get used to the notation

Thm1 $\boxed{a, b \mid P \wedge a \mid b \Leftrightarrow P = ab}$

Proof:
Given $a, b \mid P \wedge a \mid b$

$a \mid b \Rightarrow ab$ is an involution
$\Rightarrow \exists Q : Q = ab$ (def of a point)
$\Rightarrow \exists Q : Q = ab$
$\Rightarrow Q \mid a, b$

$\Rightarrow P = Q \vee a = b$

If $a = b \Rightarrow ab = 1$ but 1 is not an involution

$\Rightarrow a \nmid b$

$\Rightarrow P = Q$

$\therefore P = ab$

Given $P = ab$

$\Rightarrow a \mid b \wedge a, b \mid P$

Thm2 $\boxed{a, b, c \text{ are pairwise perpendicular } \Leftrightarrow abc = 1}$

Proof:

Given a, b, c are pairwise perpendicular,

if $abc \neq 1$

$(abc)^2 = abcabc$

$= abacbc$ (as $c \mid a \Rightarrow ca$ is an involution $\Rightarrow ac = ca \wedge a \neq c$ (iii))

$= abaccb = aabccb = 1b1b = bb = 1$ (contradiction)

as we are assuming $abc \neq 1$ this means that abc is an involution.

$\Rightarrow c \mid ab$ (as ab is an involution, and also c and abc)

as $b \mid c, ab \Rightarrow b = cab \Rightarrow 1 = ca \Rightarrow c = a$ which is a contradiction as

$c \mid a \Rightarrow \exists X : X = ca \wedge X$ is an involution $\Rightarrow X = aa = 1$. Involutions can't be the identity by definition. $\therefore abc = 1$

Given $abc = 1$, then

$ab = c \Rightarrow a \mid b$

$ab = c \Rightarrow b = ac \Rightarrow a \mid c$

$bc = a \Rightarrow b \mid c$

Thm3 $\boxed{(\forall g, P) \exists h : g \mid h \wedge P \mid h \text{ (existence of perpendiculars)}}$

There are two possibilities for P, either $P \mid g$ or $P \nmid g$.

Case 1: $P \mid g$

$\Rightarrow \exists a, b : P = ab \Rightarrow a, b, g \mid P$

$\Rightarrow \exists h : abg = h \wedge h \mid P$ (Ax3)

$\Rightarrow Pg = h \Rightarrow P = hg$

$\Rightarrow h \mid g \wedge h \mid P$

As $P \mid g \Leftrightarrow Pg = gP \wedge P \neq g \Leftrightarrow P^g = P \wedge P \neq g$ taking the negation of this means $P \nmid g \Leftrightarrow P^g \neq P \vee P = g$, this results in two cases when $P \nmid g$.

Case 2: $P^g \neq P$

$\Rightarrow \exists h : h \mid P, P^g$ (Ax1)

$\Rightarrow h^g \mid P^g, P^{gg} \Rightarrow h^g \mid P^g, P$ (remark about transformations)

$\Rightarrow h^g, h \mid P^g, P \Rightarrow h^g = h \vee P^g = P$ (Ax2)

as $P^g \neq P$ implies $h^g = h \Rightarrow gh = hg$

$P \nmid g \wedge P \mid h \Rightarrow g \neq h$

therefore by (iii) $g \mid h$

$\Rightarrow h \mid g \wedge h \mid P$

Case 3: $P = g$

$\Rightarrow \exists a, h : P = ah$

$\Rightarrow P \mid h \Rightarrow g \mid h$

$\Rightarrow h \mid g \wedge h \mid P$

Notice that Thm3 says nothing about uniqueness of perpendiculars. You might think this is a bad thing, but, in elliptic geometry, for every line, there is a point such that, any line passing through that point will be perpendicular to the line.

AxD is a cunning little axiom, it uses nothing more than an existence quantifier to say that all lines contain at least three points. To demonstrate this we need Thm5.

Thm5 | all lines have at least three points on them. |

Proof:
by AxD $(\exists g, h, j) : (g \mid h) \wedge (j \nmid g) \wedge (j \nmid h) \wedge (j \nmid gh)$.

(x) Given three distinct points P_1, P_2, P_3 on a line a, and any other line b such that $a \nmid b$, by Thm2 $\exists y_1, y_2, y_3 : y_1 \mid P_1, b \wedge y_2 \mid P_2, b \wedge y_3 \mid P_3, b$, which in turn by the definition of a point implies $\exists Q_1, Q_2, Q_3 : Q_1 = y_1 b \wedge Q_2 = y_2 b \wedge Q_3 = y_3 b$. If there isn't three points distinct on line b, then, without loss of generality $Q_1 = Q_2 \Rightarrow y_1 = y_2$. We know that $P_1 \mid y_1 \wedge P_2 \mid y_2 \wedge P_1, P_2 \mid a$ so $P_1, P_2 \mid y_1, a \Rightarrow P_1 = P_2 \vee y_1 = a$ by Ax2. As $P_1 \neq P_2 \Rightarrow a = y_1$, but we know that $b \mid y_1 \Rightarrow b \mid a$ (contradiction)
therefore there are three distinct points on b.

(xi) Take any line k, then k is not perpendicular to all g, h and j.
If it was, then $a \mid g \wedge a \mid h \wedge g \mid h$ which implies by Thm2 that $ahg = 1 \Rightarrow a = gh$. But, as $a \mid j \Rightarrow gh \mid j$ (contradicts AxD), therefore k is not perpendicular to one of g, h, j.

For this part of the proof refer to fig3.
as $g \mid h \Rightarrow \exists P : P = gh$. (def of a point)
By Thm3 $\exists a : a \mid P \wedge a \mid j$

$\Rightarrow \exists Q : Q = aj$

If $P = Q$

as $Q \mid j \Rightarrow P \mid j \Rightarrow gh \mid j$ (contradicts AxD)

$\therefore P \neq Q$

By Thm3 $\exists b : b \mid Q \wedge b \mid h$

$\Rightarrow \exists R : R = bh$

If $Q = R$

as $R \mid h \Rightarrow Q \mid h$

we know already, $Q \mid a \wedge P \mid a, h$

$\Rightarrow P = Q \vee a = h$ (Ax2)

but, from above $P \neq Q$ which implies $a = h$

as $j \mid a \Rightarrow j \mid h$ (contradicts AxD)

$\therefore Q \neq R$

If $P = R$

as $P \mid a \Rightarrow R \mid a$

we know already, $Q \mid a, b \wedge R \mid b$

$\Rightarrow Q = R \vee a = b$ (Ax2)

but, from above $Q \neq R$ which implies $a = b$

$b \mid h \Rightarrow a \mid h$

$P \mid a, h \wedge a \mid h \Rightarrow P = ah$ (Thm1)

but already we have $P = gh$

$\Rightarrow a = g$

$\Rightarrow Q = aj = gj$ (as $a = g$)

$\Rightarrow g \mid j$ (contradicts AxD)

$\therefore P \neq R$

By Thm3 $\exists x : x \mid R \wedge x \mid a$

$\Rightarrow \exists T : T = ax$

If $T = Q$

$\Rightarrow ax = aj \Rightarrow x = j$

so we have, $R, Q \mid j, b$

$\Rightarrow R = Q \vee j = b$ (Ax2)

but, from above $R \neq Q$ which implies $j = b$

as $h \mid b \Rightarrow h \mid j$ (contradicts AxD)

$\therefore T \neq Q$

If $T = P$

as $T \mid x \Rightarrow P \mid x$

we know already, $P \mid h \wedge R \mid x, h$

$\Rightarrow R = P \vee x = h$ (Ax2)

but, from above $R \neq P$ which implies $x = h$

$T = P = ax = gh \Rightarrow a = g$

as $j \mid a \Rightarrow j \mid g$ (contradicts AxD)

$\therefore T \neq P$

this now means we have three points P, T, Q all on line a that are distinct.

By (xi) line a is not perpendicular to one of g,h,j. Say it is h, then by (x) h has three distinct points on it. j Is not perpendicular to h therefore by (x) j has three distinct points on it. g is not perpendicular to j so g also has three distinct points on it by (x). Now, take any line in the group, say k, then by (xi) it is not perpendicular to one of g,h,j, this means by (x) that k also has three distinct points on it.

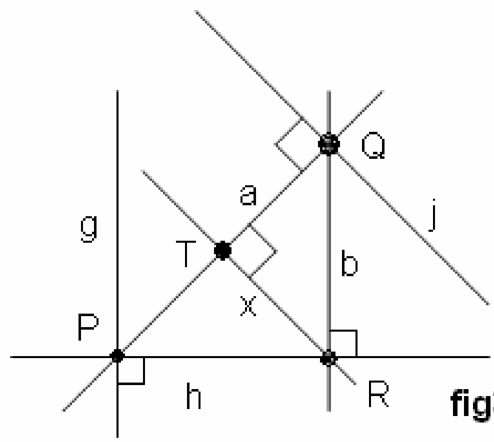


fig3

Pencils:
A pencil is determined by two lines, it is denoted as $G(ab)$, where a and b are the lines that determine it. A line c is said to be in the pencil if abc is a line, and symbolically denoted as $c \in G(ab)$. A pencil is a set of lines.

Things to notice about pencils:
given the pencil $G(ab)$, then,

$abb = a$ is a line, so, $a \in G(ab)$

$aba = b^a$ is a line, so, $b \in G(ab)$

$c \in G(ab) \Leftrightarrow abc = y \Leftrightarrow yab = c \Leftrightarrow y = cba \Leftrightarrow y = cbacc \Leftrightarrow y = (bac)^c \Leftrightarrow bac = x \Leftrightarrow c \in G(ba)$

$c \in G(ab) \Leftrightarrow abc = y \Leftrightarrow (abc)^a = x \Leftrightarrow bca = x \Leftrightarrow a \in G(bc) \Leftrightarrow a \in G(cb)$.

The upshot of this is when you see something like $a'bc'$ you know instantly that $c' \in G(a'b)$, $c' \in G(ba')$, $b \in G(c'a')$, $b \in G(a'c')$, etc.

Not all pencils have a point or perpendicular in common, such pencils are called pencils without carrier.

Thm13 $\boxed{(\forall P, a', c')\exists b \in G(a'c') \wedge b \mid P}$

Proof:

If $a' = c'$, take any line through P, say b then $a'c'b = a'a'b = b \in G(a'c')$, and we already know that $b \mid P$

If $a' \neq c'$, by Thm3 it's possible to drop perpendiculars from P to a' and c', do this and call these lines respectively a and c. As $a \mid a'$ and $c \mid c'$, let $A = aa' \wedge C = cc'$.

If $A = C$, then $c' \mid C \Rightarrow c' \mid A$, this means that $\exists y : y = c'a'a$ therefore $a \in G(a'c') \wedge a \mid P$.

For this part of the proof refer to fig4.
If $A \neq C$, then by axioms one and two it's possible to join the points A and C together with a unique line. Call this line v, from the construction of this line note that $A, C \mid v$.
Now by Thm3 $\exists d : d \mid P \wedge d \mid v$ .Then by Thm3 it implies that
$\exists a^*, c^* : A = a^* v \wedge C = c^* v$.
as $a, d, c \mid P \Rightarrow \exists b : b = adc \Rightarrow d = abc$ (Ax3)
as $a^*, d, c^* \mid v \Rightarrow a^* dc^*$ is a line.
now, $a'bc' = a'aabccc' = AabcC = AdC = va^* dc^* v = (a^* dc^*)^v$, but as $a^*, d, c^* \mid v \Rightarrow a^* dc^*$ is a line, this implies that $a'bc'$ is also a line. Therefore $b \in G(a'c') \wedge b \mid P$.

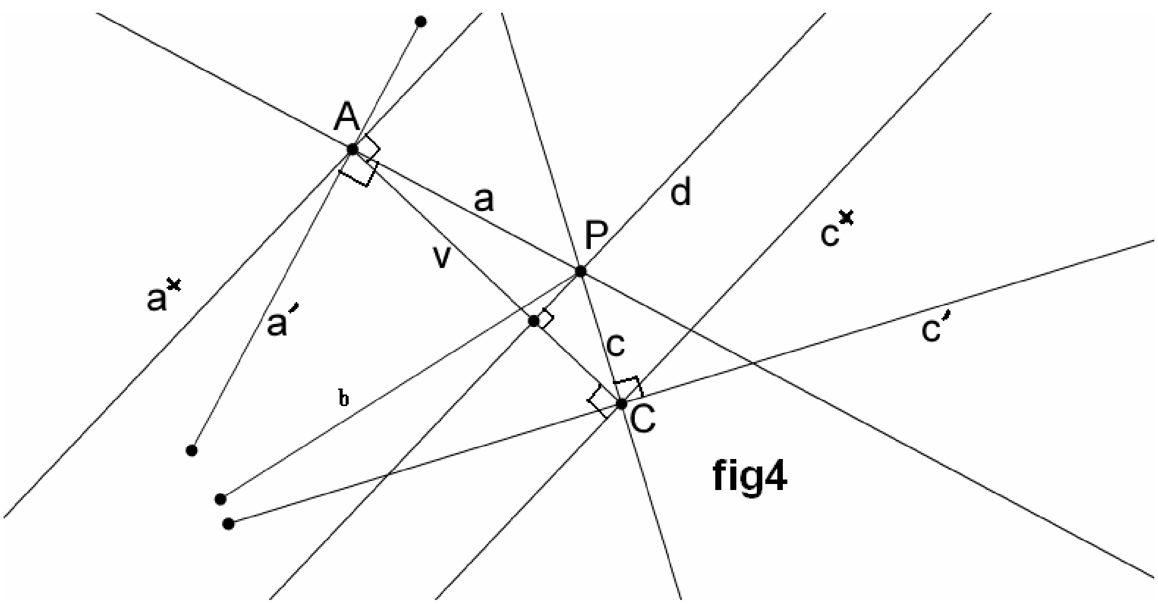


fig4

Theorem 13 allows us to find a line in a particular pencil, through any point we want. For us, it's practical application, is to allow us to prove the reduction theorem, which is a very splendid thing indeed. It states that every product of an even number of lines, is equal to the product of two lines, and every product of an odd number of lines, is equal to the product of a line and a point.

RT	Reduction theorem

(RTi) $(\forall u,v,W)(\exists a,b): uvW = ab$

Proof:

$\exists g : g \mid W \wedge g \in G(uv)$ by Thm13

$g \in G(uv) \Rightarrow \exists a : uvg = a$ (def of pencil)

$g \mid W \Rightarrow \exists b : b = gW$ by Thm3

$\Rightarrow uvW = uvggW = ab$

(RTii) $(\forall u,v,w)(\exists a,B): uvw = aB$

Proof:

$\exists U : U \mid u$ Thm5

$\exists g : g \mid U \wedge g \in G(vw)$ by Thm13

$g \in G(vw) \Rightarrow w \in G(gv)$ ("things to notice about pencils")

$w \in G(gv) \Rightarrow \exists h : gvw = h$ (def of pencil)

$\Rightarrow vw = gh$

$\exists b : b \mid U \wedge b \mid h$ by Thm3

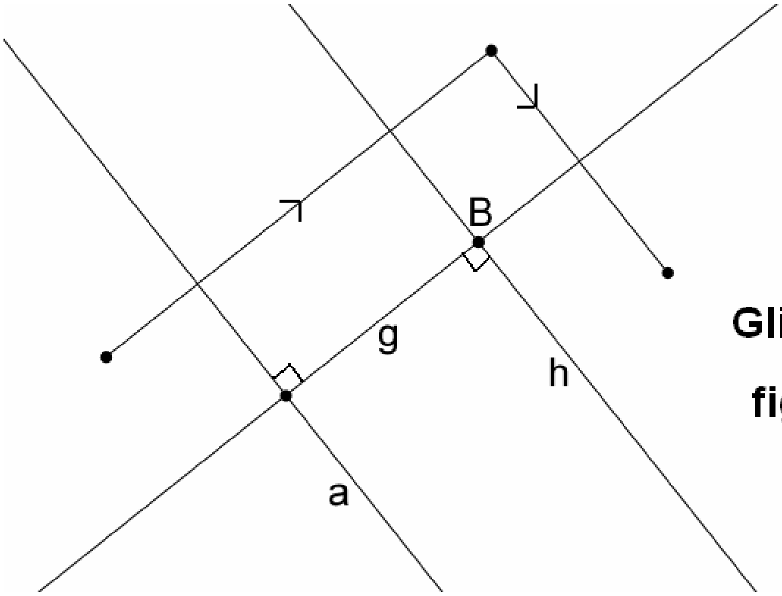
$u,g,b \mid U \Rightarrow \exists a : a = ugb$ (Ax3)

$b \mid h \Rightarrow \exists B : B = bh$ (def of a point)

$\Rightarrow uvw = ugh = ugb.bh = aB$

Now, given any element $\beta \in M$, that is a sequence of two or more line reflections, by induction we see that using RTi and RTii allows us to write it as either, aB or ab. If, the element is only of length one, say c, then of course this can be written as, $c = ccc = aB$, by RTii. If the element is of length zero, well who knows. In real life, the reduction theorem says, that you need not flip a piece of paper over more than three times, to move it anywhere you want to. A corollary to the reduction theorem is that, an involution is either a point or a line. This is so, because all involutions in the group look like ab or aB, ab is a point by definition, while aB is a line by Thm3.

In Euclidean geometry, (the one we live in?), we saw from the motivation for the axioms, that something of the form ab was a rotation if the two lines have a point in common, and a translation otherwise. We didn't think at all about what aB meant, except in the case when $a \mid B$, we were unaware of this though, but given $a \mid B$ implies $\exists b : B = ab$ we see that $aB = aab = b$ which is just a reflection. This means there's just one case to go, what do the isomorphisms look like when $a \nmid B$? These ones we haven't met to date, these are called glide reflections. I imagine these ones like satellites in space, they rotate around their own axis while moving forward. (refer to fig5) To see this, given a glide reflection aB we can always find, by using Thm3 two times, g and h such that $B = hg \wedge g \mid h,a$, this implies that h is parallel to a. This means that $aB = ah.g$. Now we have a translation ah then a line reflection g perpendicular to the direction of translation. So, by the reduction theorem the only isometries in the Euclidean world are rotations, translations, reflections, and glide reflections. This is something people may have known but few would have seen. By using Bachmann groups it's quite provable.



Glide reflection

fig5

Summary:

Looking at geometry in a group theoretic way is a very interesting take on geometry. Some things using this technique are much easier than using traditional coordinates or something, while others are far more difficult. Very few English books have been written about Bachmann groups. There are only four books that I know about, so recommended reading is sparse, the best of these is "H. Behnke et al. (eds), *fundamentals of mathematics*; volume II: geometry, M.I.T Press, 1974.". Others are "M. Henle, *Modern Geometries*, Prentice Hall, 1997, 2001", and "G. Ewald, *Geometry: An Introduction*, Wadsworth, 1971". After only knowing about Bachmann groups for a few months, I have had my question proved as to why there are only four different types of isometries. I have heard others describe geometry as beautiful before, and after seeing Bachmann groups, I agree.

Jonathan Olds.